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Functional Calculus and Duality for Closed Operators

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In this paper, we continue our spectral-theoretic study [8] of unbounded closed operators in the framework of the spectral decomposition property and decomposable operators. Given a closed operator T with nonempty resolvent set, let $f \to f(T)$ be the homomorphism of the functional calculus. We show that if T has the spectral decomposition property, then $f(T)$ is decomposable. Conversely, if f is nonconstant on every component of its domain which intersects the spectrum of T, then $f(T)$ decomposable implies that T has the spectral decomposition property. A spectral duality theorems follows as a corollary. Furthermore, we obtain an analytic-type property for the canonical embedding J of the underlying Banach space X into its second dual X^{**}.

1. INTRODUCTION

We refer to [8] for notations and terminology, but for convenience we repeat some of the basic definitions. Throughout this paper, T is an unbounded closed operator with domain D_T and range R_T in an abstract Banach space X over the complex field $\mathbb{C}$. $\mathbb{C}_\infty$ denotes the one-point compactification of $\mathbb{C}$. For $S \subset \mathbb{C}$, we assume that every finite open cover of S, in symbols $\{G_i\}_{i=0}^{n} \in \operatorname{cov} S$ has at most one unbounded set G_0. An open

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$G \subset \mathbb{C}$ is a neighborhood of ∞, in symbols $G \in V_\infty$, if its complement G^c is compact in $\mathbb{C}$. If T has the single-valued extension property (SVEP) then, for $S \subset \mathbb{C}$,

$$X(T, S) = \{x \in X: \sigma_T(x) \subset S\}$$

is the corresponding spectral manifold.

Inv T represents the lattice of all subspaces of X which are invariant under T. For $Y \in \operatorname{Inv} T$, $T| Y$ is the restriction of T to Y and T/Y denotes the coinduced operator on the quotient space X/Y. For a set $Z \subset X$, $Z^\perp$ is the annihilator of Z in X^* and, for $Z \subset X^*$, ${}^\perp Z$ denotes the preannihilator of Z in X. A_T denotes the family of all $\mathbb{C}$-valued functions which are locally analytic on some neighborhood Ω_f of $\sigma(T)$.

1.1. DEFINITION. T is said to have the *spectral decomposition property* (SDP) if, for every $\{G_i\}_{i=0}^n \in \operatorname{cov} \sigma(T)$ with $G_0 \in V_\infty$, there exists a system $\{Y_i\}_{i=0}^n \subset \operatorname{Inv} T$ satisfying the following conditions:

(I) $Y_i \subset D_T$ if G_i is relatively compact $(1 \leqslant i \leqslant n)$;

(II) $X = \sum_{i=0}^n Y_i$ and $\sigma(T| Y_i) \subset G_i$ (or $\sigma(T| Y_i) \subset \bar{G}_i$), $0 \leqslant i \leqslant n$.

There are two special types of invariant subspaces which occur frequently in spectral decompositions.

1.2. DEFINITION [4]. Given T, $Y \in \operatorname{Inv} T$ is a *spectral maximal space* of T if, for every $Z \in \operatorname{Inv} T$, the inclusion $\sigma(T|Z) \subset \sigma(T| Y)$ implies $Z \subset Y$.

1.3. DEFINITION [5]. Given T, $Y \in \operatorname{Inv} T$ is said to be *analytically invariant* under T if, for every function $f: \omega_f \to D_T$ analytic on an open $\omega_f \subset \mathbb{C}$, the condition $(\lambda - T) f(\lambda) \in Y$ on ω_f implies $f(\lambda) \in Y$ on ω_f.

The spectral maximal space is instrumental in the definition of the decomposable operator concept.

1.4. DEFINITION [4]. T is said to be *decomposable* if, for any $\{G_i\}_{i=0}^n \in \operatorname{cov} \sigma(T)$ with $G_0 \in V_\infty$, there is a system $\{Y_i\}_{i=0}^n$ of spectral maximal spaces of T satisfying conditions (I) and (II) of Definition 1.1.

A useful characterization of an analytically invariant subspace Y involves the coinduced operator T/Y on the quotient space X/Y.

1.5. PROPOSITION [5]. *Given T, let $Y \in \operatorname{Inv} T$ be such that $Y \subset D_T$. Then Y is analytically invariant under T iff T/Y has the* SVEP.

If T has the SDP then, for closed $F \subset \mathbb{C}_\infty$, $X(T, F)$ is a spectral maximal space of T. For compact $F \subset \mathbb{C}$, a direct-sum decomposition of $X(T, F)$ is available.

1.6. PROPOSITION [8]. *Given T with the* SDP, *for every compact $F \subset \mathbb{C}$, there is $\Xi(T, F) \in \operatorname{Inv} T$ satisfying properties*:

$$X(T, F) = \Xi(T, F) \oplus X(T, \varnothing),$$
$$\sigma[T \mid \Xi(T, F)] = \sigma[T \mid X(T, F)].$$

The subspace $\Xi(T, F)$, referred to as a *T-bounded spectral maximal space*, is characterized by the implication

$$Z \in \operatorname{Inv} T,\ Z \subset D_T,\ \sigma(T \mid Z) \subset \sigma[T \mid \Xi(T, F)] \Rightarrow Z \subset \Xi(T, F).$$

Moreover, both $X(T, F)$ and $\Xi(T, F)$ are hyperinvariant under T (i.e., invariant under every bounded linear operator which commutes with T) [ibid.].

In terms of spectral and T-bounded spectral maximal spaces, the spectral decomposition of T with the SDP takes the following form

$$X = X(T, \bar{G}_0) + \sum_{i=1}^{n} \Xi(T, \bar{G}_i), \tag{1.1}$$

where $\{G_i\}_{i=0}^n \in \operatorname{cov} \sigma(T)$ with $G_0 \in V_\infty$.

While the classes of operators with the SDP and decomposable operators are indistinguishable in the Banach algebra $B(X)$ [6], their unbounded extensions no longer coincide. They do iff $X(T, \varnothing) = \{0\}$, [8].

2. FUNCTIONAL CALCULUS

Throughout this section we assume that $\rho(T) \neq \varnothing$. Given T, fix $\alpha \in \rho(T)$. Define the map $\Phi: \mathbb{C}_\infty \to \mathbb{C}_\infty$ by

$$\Phi(\lambda) = \begin{cases} (\lambda - \alpha)^{-1}, & \lambda \neq \alpha,\ \lambda \neq \infty, \\ 0, & \lambda = \infty, \\ \infty, & \lambda = \alpha. \end{cases}$$

Denote $A = (T - \alpha)^{-1}$.

2.1. LEMMA. *If A is decomposable, then*

(i) *for every closed* $F \subset \mathbb{C}$, $X(A, F) \in \operatorname{Inv} T$;

(ii) *if* $0 \notin F$, *then* $X(A, F) \subset D_T$.

Proof. We divide the proof into three parts.

Part I. In this part we show that for closed $F \subset \mathbb{C}$ with $0 \notin F$, we have

$$X(A, F) \in \operatorname{Inv} T \quad \text{and} \quad X(A, F) \subset D_T. \tag{2.1}$$

If follows from $0 \notin F$ that $[A \mid X(A, F)]^{-1}$ is bounded on $X(A, F)$. Let $x \in X(A, F)$. Then

$$x = A[A \mid X(A, F)]^{-1} x \in R_A = D_T.$$

Since A is injective, it follows from

$$A\{(T-\alpha)x - [A \mid X(A, F)]^{-1} x\} = x - x = 0$$

that

$$(T-\alpha)x = [A \mid X(A, F)]^{-1} x \in X(A, F) \tag{2.2}$$

and hence $X(A, F) \in \operatorname{Inv} T$. Thus, (2.1) is proved.

Part II. Let $G \subset \mathbb{C}$ be open with $0 \in G$. Put $G_0 = G$ and choose $G_1 \subset \mathbb{C}$ open such that $\{G_0, G_1\} \in \operatorname{cov} \sigma(A)$ and $0 \notin \bar{G}_1$. Since A is decomposable, we have

$$X = X(A, \bar{G}_0) + X(A, \bar{G}_1). \tag{2.3}$$

Let $x \in X(A, \bar{G}_0) \cap D_T$. In view of (2.3), there is a representation

$$(T-\alpha)x = y_0 + y_1 \text{ with } y_i \in X(A, \bar{G}_i),\ i = 0, 1.$$

Let $z_1 = Ay_1$. Then $z_1 \in X(A, \bar{G}_1)$, furthermore

$$z_1 = A[(T-\alpha)x - y_0] = x - Ay_0 \in X(A, \bar{G}_0)$$

and hence

$$z_1 \in X(A, \bar{G}_0) \cap X(A, \bar{G}_1) = X(A, \bar{G}_0 \cap \bar{G}_1). \tag{2.4}$$

Since $0 \notin \bar{G}_0 \cap \bar{G}_1$, it follows from Part I that $X(A, \bar{G}_0 \cap \bar{G}_1) \in \operatorname{Inv} T$. Consequently, $y_1 = (T-\alpha)z_1 \in X(A, \bar{G}_0 \cap \bar{G}_1) \subset X(A, \bar{G}_0)$ implies

$$(T-\alpha)x = y_0 + y_1 \in X(A, \bar{G}_0)$$

and hence $X(A, \bar{G}_0) \in \operatorname{Inv} T$.

Part III. Let $F \subset \mathbb{C}$ be closed with $0 \in F$. Choose open sets $\{G_n\}_{n=1}^{\infty}$ such that $F \subset G_n$ for all n and $F = \bigcap_{n=1}^{\infty} \bar{G}_n$. It follows from

$$X(A, F) = \bigcap_{n=1}^{\infty} X(A, \bar{G}_n)$$

and from Part II, that $X(A, F) \in \operatorname{Inv} T$. ∎

2.2. THEOREM. *T has the* SDP *iff A is decomposable.*

Proof. (Only if) Assume that T has the SDP. Let $\{G_i\}_{i=0}^{n} \in \operatorname{cov} \sigma(A)$ and, without loss of generality, suppose that $0 \in G_0$ and $0 \notin \bar{G}_i$ $(1 \leqslant i \leqslant n)$. Put $H_i = \Phi^{-1}(G_i)$, $0 \leqslant i \leqslant n$, and note that $H_0 \in V_\infty$ and each H_i $(1 \leqslant i \leqslant n)$ is relatively compact. In view of $\sigma(A) = \Phi[\sigma(T) \cup \{\infty\}]$, (e.g., [3, Lemma VII.9.2]), $\{H_i\}_{i=0}^{n}$ is an open cover of $\sigma(T)$. By the SDP, (1.1),

$$X = X(T, \bar{H}_0) + \sum_{i=1}^{n} \Xi(T, \bar{H}_i). \tag{2.5}$$

Since the subspaces $X(T, \bar{H}_0)$, $\Xi(T, \bar{H}_i)$ $(1 \leqslant i \leqslant n)$ are hyperinvariant under T, they are invariant under A. By using the above quoted spectral mapping theorem, we obtain

$$\sigma[A \mid X(T, \bar{H}_0)] = \Phi(\sigma[T \mid X(T, \bar{H}_0)] \cup \{\infty\}) \subset \Phi(\bar{H}_0) = \bar{G}_0, \tag{2.6}$$

$$\sigma[A \mid \Xi(T, \bar{H}_i)] = \Phi\{\sigma[T \mid \Xi(T, \bar{H}_i)]\} \subset \Phi(\bar{H}_i) = \bar{G}_i, \quad 1 \leqslant i \leqslant n. \tag{2.7}$$

By (2.5), (2.6), and (2.7), A has the SDP and since A is bounded, it is decomposable.

(If) Assume that A is decomposable. Let $\{H_i\}_{i=0}^{n} \in \operatorname{cov} \sigma(T)$ with $H_0 \in V_\infty$. Put $G_i = \Phi(H_i)$, $0 \leqslant i \leqslant n$, and note that $\{G_i\}_{i=0}^{n} \in \operatorname{cov} \sigma(A)$.

Since A is decomposable, we have

$$X = \sum_{i=0}^{n} X(A, \bar{G}_i).$$

It follows from Lemma 2.1 that $X(A, \bar{G}_i) \in \operatorname{Inv} T$, $0 \leqslant i \leqslant n$ and, for $1 \leqslant i \leqslant n$, $T \mid X(A, \bar{G}_i)$ is bounded. Furthermore, we have

$$\sigma[T \mid X(A, \bar{G}_i)] = \Phi^{-1}\{\sigma[A \mid X(A, \bar{G}_i)]\} \subset \Phi^{-1}(\bar{G}_i) = \bar{H}_i, \qquad 0 \leqslant i \leqslant n.$$

Consequently, T has the SDP. ∎

2.3. COROLLARY. *If T has the* SDP *then, for every $f \in A_T$, $f(T)$ is decomposable. Conversely, let $f \in A_T$ be nonconstant on every component of* *its domain which intersects $\sigma(T)$. If $f(T)$ is decomposable then T has the* SDP.

Proof. First, assume that T has the SDP. Given $f \in A_T$, let $g(\mu) = f[\Phi^{-1}(\mu)]$. By the functional calculus,

$$g(A) = f(T). \tag{2.8}$$

By Theorem 2.2, A is decomposable and hence $g(A)$ is decomposable by [2, Theorem 2.11]. Thus, $f(T)$ is decomposable.

Next, assume that $f \in A_T$ is nonconstant on every component of Ω_f which intersects $\sigma(T)$ and let $f(T)$ be decomposable. It follows from [1, Theorem 3.3] and (2.8) that A is decomposable. Thus, T has the SDP, by Theorem 2.2. ∎

2.4. Corollary. *Let T be densely defined. Then T has the* SDP *iff T^* has the* SDP.

Proof. T has the SDP iff A is decomposable, iff A^* is decomposable [9], iff T^* has the SDP. ∎

A different approach to the functional calculus on unbounded decomposable operators is to be found in [7].

3. A Property in Duality

In this section we waive the restriction $\rho(T) \neq \varnothing$ but we shall avail ourselves of a pertinent domain-density condition:

$$(*): \quad \bar{D}_T = X,\ \bar{D}_{T^*} = X^*;$$

$$(**): \quad (*) \text{ and } \bar{D}_{T^{**}} = X^{**};$$

$$(***): \quad (**) \text{ and } \bar{D}_{T^{***}} = X^{***}.$$

With J and K the canonical embeddings of X into X^{**} and of X^* into X^{***}, respectively, the following direct-sum decomposition holds (e.g. [9]):

$$X^{***} = KX^* \oplus (JX)^\perp.$$

Then, if P is the projection of X^{***} onto KX^* along $(JX)^\perp$, we have

$$(X^{**}/JX)^* = (JX)^\perp = N(P); \tag{3.1}$$

$$(X^{***}/KX^*)^* = (KX^*)^\perp = N(P^*), \tag{3.2}$$

where $N(\cdot)$ denotes the null space. Furthermore [8], under (**), P commutes with T^{***} and if (***) holds then P^* commutes with T^{****}.

3.1. LEMMA [8, Lemma 2.2]. *Given T, let $Y \in \operatorname{Inv} T$ be such that $\overline{Y \cap D_T} = Y$. If $T^* \mid Y^\perp$ is densely defined, then T/Y is closable and $(T/Y)^* = T^* \mid Y^\perp$.*

3.2. PROPOSITION [8, Theorem 2.7]. *Given T, assume that (***) holds. Then $JD_T = JX \cap D_{T^{**}}$, $T^{***} \mid (JX)^\perp$ is densely defined and, for every $Jx \in JD_T$, we have*

$$JTx = T^{**}Jx.$$

Now, we are in a position to prove an extension of Proposition 1.5 in duality.

3.3. THEOREM. *Given T, assume that (***) holds. If T^{****} has the* SVEP, *then JX is analytically invariant under T^{**}.*

Proof. First, we note that $(X^{**}/JX)^{**}$ and $N(P^*)$ are topologically isomorphic. In fact, X^{***}/KX^* and $N(P)$ are topologically isomorphic and our assertion follows from (3.1) and (3.2).

Next, we show that T^{**}/JX is closable, and $(T^{**}/JX)^{**}$ is similar to $T^{****} \mid N(P^*)$. By Proposition 3.2, JX is invariant under T^{**} and hence T^{**}/JX can be defined. Also by Proposition 3.2, $T^{***} \mid N(P)$ is densely defined and then Lemma 3.1 implies that T^{**}/JX is closable and $(T^{**}/JX)^* = T^{***} \mid N(P)$. The similarity of $T^{**}/JX)^{**}$ and $T^{****} \mid N(P^*)$ will follow from the similarity of $(T^{***})\hat{} = T^{***}/KX^*$ and $V^{***} = T^{***} \mid NP$. Let $A\colon X^{***}/KX^* \to N(P)$ be the topological isomorphism and let $(x^{***})\hat{} = x^{***} + KX^* \in X^{***}/KX^*$. Let $x^{***} \in D_{T^{***}}$. Then $(x^{***})\hat{} \in D_{(T^{***})\hat{}}$ and since P commutes with T^{***}, we have $(I-P)\,x^{***} \in D_{V^{***}}$. Then $A(x^{***})\hat{} = (I-P)\,x^{***}$ implies

$$AD_{(T^{***})\hat{}} \subset D_{V^{***}}.$$

Conversely, let $x^{***} \in D_{V^{***}}$. Then $x^{***} \in D_{T^{***}} \cap N(P)$, furthermore $(x^{***})\hat{} \in D_{(T^{***})\hat{}}$ and $A(x^{***})\hat{} = x^{***}$. Consequently, we have

$$AD_{(T^{***})\hat{}} = D_{V^{***}}. \tag{3.3}$$

Now, letting $x^{***} \in D_{T^{***}}$, it follows from (3.3) and from

$$\begin{aligned} A(T^{***})\hat{}(x^{***})\hat{} &= A(T^{***}x^{***})\hat{} = (I-P)\,T^{***}x^{***} \\ &= T^{***}(I-P)\,x^{***} = V^{***}A(x^{***})\hat{}, \end{aligned}$$

that $(T^{***})\hat{}$ is similar to V^{***}. Consequently, $(T^{***})\hat{}$ is closed. Furthermore, (3.2) implies

$$[(T^{***})\hat{}]^* = T^{****} \mid N(P^*)$$

and hence it follows that $[T^{***} \mid N(P)]^*$ and $T^{****} \mid N(P^*)$ are similar. Thus, $(T^{**}/JX)^{**}$ and $T^{****} \mid N(P^*)$ are similar.

Now let $f^{**}: \omega \to D_{T^{**}}$ be analytic on an open $\omega \subset \mathbb{C}$ and satisfy condition

$$(\lambda - T^{**}) f^{**}(\lambda) \in JX \qquad \text{on } \omega. \tag{3.4}$$

On the quotient space X^{**}/JX, with $(f^{**})\hat{}$ corresponding to f^{**}, (3.4) gives rise to

$$(\lambda - T^{**}/JX)(f^{**})\hat{}(\lambda) = \hat{0} \qquad \text{on } \omega. \tag{3.5}$$

By hypothesis, $T^{****} \mid N(P^*)$ has the SVEP and hence $(T^{**}/JX)^{**}$ inherits the same property, by similarity. Thus, the minimal closed extension $\overline{T^{**}/JX}$ of T^{**}/JX has the SVEP and (3.5) implies that $(f^{**})\hat{}(\lambda) = \hat{0}$ on ω or, equivalently, $f^{**}(\lambda) \in JX$ on ω. ∎

3.4. Corollary. *Given T, assume that* $(***)$ *holds. If T^{****} has the* SVEP *then, for any $E \subset \mathbb{C}$, we have*

$$JX(T, E) = X^{**}(T^{**}, E) \cap JX.$$

Proof. Assume that T^{****} has the SVEP. Then T^{**} as well as T has the SVEP. Consequently, the spectral manifolds $X^{**}(T^{**}, E)$ and $X(T, E)$ are defined. The inclusion

$$JX(T, E) \subset X^{**}(T^{**}, E) \cap JX \tag{3.6}$$

is evident. To prove the opposite inclusion, let $Jx \in X^{**}(T^{**}, E) \cap JX$. The resolvent function $x^{**}(\cdot)$ of Jx identically verifies equation

$$(\lambda - T^{**})\, x^{**}(\lambda) = Jx \qquad \text{on } \rho_{T^{**}}(Jx).$$

Since, by Theorem 3.3, JX is analytically invariant under T^{**}, for every $\lambda \in \rho_{T^{**}}(Jx)$, we have $x^{**}(\lambda) \in JX$. For $\lambda \in \rho_{T^{**}}(Jx)$, let $f(\lambda) = J^{-1}x^{**}(\lambda)$. Then $f: \rho_{T^{**}}(Jx) \to D_T$ is analytic on $\rho_{T^{**}}(Jx)$ and

$$(\lambda - T) f(\lambda) = (\lambda - T) J^{-1}x^{**}(\lambda) = J^{-1}(\lambda - T^{**})\, x^{**}(\lambda) = J^{-1}Jx = x.$$

Thus, $\rho_T(x) \supset \rho_{T^{**}}(Jx)$ or, equivalently,

$$\sigma_T(x) \subset \sigma_{T^{**}}(Jx) \subset E.$$

Thus, $x \in X(T, E)$ and the opposite of (3.6) follows. ∎

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